Self-Replicating Loops

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Abstract: Self-replicating melodic loops have already been discussed in the last section of the author's book Self-Replicating Melodies*, and in a more mathematical article by David Feldman, and employed in several compositions by Johnson and other composers. This presentation is an introduction to the idea, along with computer output of orbit structures, and some other more recent observations.*

In New York in the early 70s a number of composers of my generation were reacting against the American academic musical tradition, best known in Europe in the examples of Elliott Carter and Milton Babbitt. We wanted something simpler, something that spoke more directly to a wider audience. Of course, our reaction was to become known as American minimalism, and the best known segment of it was the repetitive music of Terry Riley and Philip Glass and others, but there were other important forms: the long drone tones of La Monte Young, the meditative sounds of Pauline Oliveros, the sustained microtonal textures of Phill Niblock, the acoustical experiments of Alvin Lucier, and of course, pieces on limited scales, such as John Adams' *Shaker Loops* and my own *Four-Note Opera* (1972). As the '70s went on, however, I found myself increasingly attracted to the more rationally organized forms of minimalism, such as the logical melodies of Frederic Rzewski. I had been playing piano duets with Philip Corner, another colleague who liked to count notes and calculate logical sequences, and it was really in playing piano with him that I began to appreciate melodic loops, sometimes playing them in unison in different tempos, like this:

A	b	c	a	b	c	a	b	c	a	b	c	a	b c…
A				b				c				a…	

This is a trivial case, since any loop of n notes can be in unison with itself when played $(n + 1)$ times slower or faster. I wanted to go further, and find more interesting ways of making loops within loops, and one day I decided to consult my friend David Feldman with this question: How could I write a melodic loop of, say, 15 notes, in such a way that one will hear the same melody if one listens to every note, or to every second note? Feldman is himself a composer, but he is also a professional mathematician, and he was able to answer the question a couple of days later. He explained that I could write whatever notes I desired, as long as are were not more than five notes in the scale, and provided that they fall together on the loop in this way.

(0)
(1 2 4 8)
(3 6 9 12)
(5 10)
(7 11 13 14)

This means that note *0* can be anything, but that notes *1, 2, 4,* and *8* of the loop must be identical, notes *3, 6, 9,* and *12* must be identical, and so on. With this information I wrote the first "self-replicating melody," which was to become *Rational Melody No. 15* (1981). Sometimes we find extraordinary things and just don't see any way of going further with them until much later, and it was more than 10 years later that I began to realize how many things this principle would lead to.

I won't pretend to be able to really explain how these orbit structures come about, as this is a job for a mathematician well versed in group theory, but for musicians, let me demonstrate how one might compose a self-replicating melody without really knowing about such things. Let's say that we want to compose a seven-note loop that makes a copy of itself at *2 : 1*. Since we must be able to hear the same melody at the two different tempos simultaneously, we have to see how the seven notes of the fast loop will fall against the seven notes of the slow loop:

0 1 2 3 4 5 6 0 1 2 3 4 5 6 0…
0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0…

Note *1* of the slow melody occurs at the same time as note *2* of the fast melody, and note *2* of the slow melody occurs at the same time as note *4* of the slow melody, so it is clear that notes *1, 2,* and *4* must all be the same. Notes *3, 5,* and *6* also must be identical, since they sometimes occur simultaneously, so now we can deduce that there are three orbits in this form:

(0)
(1 2 4)
(3 5 6)

In other words, what I call "the solution loop" must be a melody on a three-note scale that has the form *0 1 1 2 1 2 2*. Here you can see how the two different speeds fall in unison if a seven-note loop employs three pitches in this way :

0 1 1 2 1 2 2 0 1 1 2 1 2 2…
0 1 1 2 1 2 2 0 1 1 2 1 2 2 0 1 1 2 1 2 2 0 1 1 2 1 2…

Naturally, if the loop at half time is in unison with the loop at the original tempo, the same loop at quarter time must be in unison with the loop at half time, and we can continue with voices 8 times slower than the original, 16 times slower, and so on, and they will all be in unison.

There is no way to construct an eight-note loop in such a way that it will self-replicate at *2 : 1*, because *8* is divisible by *2*. An eight-note loop can make a copy of itself at *3 : 1*, however, or *5 : 1*, or *7 : 1* in these ways:

3 : 1: *(0) (1 3) (2 6) (4) (5 7)*
5 : 1: *(0) (1 5) (2) (3 7) (4) (6)*
7 : 1: *(0) (1 7) (2 6) (3 5) (4)*

There are five different orbits at *3 : 1* and *7 : 1*, and six different orbits at *5 : 1*, so we can have many different notes in the scale. There is another orbit structure that I find more interesting however.

(0 6) (1 5) (2 4) (3 7)

Here there are only four orbits, but they are arranged in such a way that, if we set the starting times right, they permit our melodic loop to make copies of itself at all of the above ratios, and in fact, *9 : 1, 11 : 1* (11 modulo 8 = 3) and every other odd numbered ratio as well.

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0   1   2   3   2   1   0   3   0   1   2   3   2   1   0   3   0   1   2   3   2   1   0   3   0   1   2   3   2…
                        0           1           2           3           2           1           0           3…
0                   1                   2                   3                   2                   1…
                        0                           1                           2                           3…
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As I was studying all this, quite a few months after completing *Self-Similar Melodies*, I found it necessary to write a computer program that would compute the orbit structures of loops I might want to use someday. Good mathematicians told me that my 58 pages of computer output, which considered all the possible self-replicating loops of 36 notes or less, is completely trivial, because they can figure out the orbit structures for any given situation about as easily as we can punch out multiplication on our calculators. But for me the exercise was not at all a waste of time, and I often refer to my 58 pages of orbit structures when I am looking for one that has particular characteristics. Let us look at all the possible self-replicating loops of 13 notes, for example:

Loop of 13, ratio of 2
(0)
(1 2 4 8 3 6 12 11 9 5 10 7)

Loop of 13, ratio of 3
(0)
(1 3 9)
(2 6 5)
(4 12 10)
*(*7 8 11)

Loop of 13, ratio of 4
(0)
(1 4 3 12 9 10)
(2 8 6 11 5 7)

Loop of 13, ratio of 5
(0)

(1 5 12 8)
(2 10 11 3)
(4 7 9 6)

Loop of 13, ratio of 6
(0)
(1 6 10 8 9 2 12 7 3 5 4 11)

Loop of 13, ratio of 7
(0)
(1 7 10 5 9 11 12 6 3 8 4 2)

Loop of 13, ratio of 8
(0)
(1 8 12 5)
(2 3 11 10)
(4 6 9 7)

Loop of 13, ratio of 9
(0)
(1 9 3)
(2 5 6)
(4 10 12)
(7 11 8)

Loop of 13, ratio of 10
(0)
(1 10 9 12 3 4)
(2 7 5 11 6 8)

Loop of 13, ratio of 11
(0)
(1 11 4 5 3 7 12 2 9 8 10 6)

Loop of 13, ratio of 12
(0)
(1 12)
(2 11)
(3 10)
(4 9)
(5 8)
(6 7)

Since *13* is a prime number, all ratios from *2* to *12* are possible, and four of these produce only two orbits, which do not permit very interesting two-note melodies. It is not surprising that the orbits are the same whether the ratio is *3 : 1* or *9 : 1*, since *9* is three times three, but note that the results are completely different when the ratio is 2 : 1 or 4 : 1 or 8 : 1. A number of other observations might be made, but let me leave you to do your own research here and conclude with something else. You may find this information useful, and probably won't, but it is a good example of how computer lists can be useful, even when mathematicians tell us that the information is trivial.

Naturally scales of only three or four notes often seem too limited, and one question that interested me ever since the beginning of this investigation was how to permit the complete chromatic. What is the smallest loop possible that permits 12 orbits, the complete chromatic scale? I posed this question to very good mathematicians like Feldman, and my friend Jean-Paul Allouche, and they couldn't tell me. It is not hard to find loops with 25 or 30 different notes, that can have 12 or more orbits, but it is difficult to say with any certainty that one has found the shortest one. Only after making my 58-page list was it clear, much the surprise of everyone, that the answer to my question was a loop much shorter than anyone expected, a loop only 16 notes long, with this orbit structure.

Loop of 16, ratio of 9
(0)
(1 9)
(2)
(3 11)
(4)
(5 13)
(6)
(7 15)
(8)
(10)
(12)
(14)

Unfortunately, the loop does its thing only at a ratio of 9 : 1, which is musically a little difficult to hear and work with, but it is information that can perhaps be useful for composers who like difficult things. It is something that a minimalist like myself can turn over to maximal composers, like Elliott Carter and Milton Babbitt.